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TORSION OF FINITE ELASTIC CYLINDERS WELDED TO CIRCULAR PLATES OR CYLINDRICAL SHELLS

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The following nonclassical problem arises in the course of determining the stress-deformation state of finite elastic bodies supported by thin-walled elastic elements (rods, plates and shells) (*) It is required to expand a given boundary function which undergoes a finite jump on the line of end section of the thin-walled element, into a series in terms of eigenfunctions. The unusual character of this problem is illustrated e.g. by the case in which the boundary function is zero everywhere except at the line of discontinuity.
Two problems of this type are solved in the present paper in closed form. Torsion of finite elastic cylinders welded end-on to circular plates of constant thickness, or welded along the lateral surfaces to cylindrical shells of constant thickness, is studied. Arbitrary fundamental conditions at the lateral boundary surface of the cylinder in the first problem and at the cylinder ends in the second problem, are satisfied exactly using the orthogonality relations derived in the paper. Similar orthogonality relations in the presence of a load were used earlier [1] in investigating oscillations of mechanical systems with concentrated masses. The problems of convergence of the solutions obtained and their behavior at the corner points are studied here for the first time. Numerical results obtained are presented.

1. Let a circular plate of constant thickness h, be welded to the end $z = 1$ of a finite elastic cylinder $0 \leqslant z \leqslant 1$, $0 \leqslant r \leqslant R$. The other end is assumed, for definiteness, to be rigidly clamped (Fig. 1a). Arbitrary axisymmetric tangential forces $f(z)$ are applied to the lateral surface, and the outer surface of the plate is load-free within the zone of contact. It is, however, acted upon by a torsional moment M along the circumference $z = 1$, $r = R$

Let us write the boundary conditions for the finite cylinder

$$v = 0 \quad (z = 0,\ 0 \leqslant r \leqslant R) \tag{1.1}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} - \frac{1}{G_0 h}\tau_{\varphi z} = 0 \quad (z = 1, 0 \leqslant r \leqslant R) \tag{1.2}$$

*) Glagovskii, V. B. and Nuller, V. M., Analogs of the Fourier and Schiff orthogonality relations on the problems for supported elastic regions. Izv. AS SSSR MTT, Seminars of the Leningrad Polytechnic Inst., № 4, 1973.

$$\tau_{r\varphi} = f(z) \quad (r = R, 0 \leqslant z < 1) \tag{1.3}$$

$$\tau_{r\varphi} = MG(2\pi R^2 G_0 h)^{-1} \quad (r = R, z = 1) \tag{1.4}$$

Here G and G_0 are the shear moduli of the materials of the plate and the cylinder, respectively. We shall seek the solution of the problem (1.1)-(1.4) in the form of a series following the system of homogeneous solutions of the problem (1.1), (1.2).

$$v = \sum_{k=1}^{\infty} v^k, \quad v^k = G^{-1} A_k p_k I_1(p_k r) \sin p_k z \quad (k = 1, 2, \ldots) \tag{1.5}$$

$$\tau_{\varphi z}{}^k = A_k p_k^2 I_1(p_k r) \cos p_k z, \quad \tau_{r\varphi}{}^k = A_k p_k^2 I_2(p_k r) \sin p_k z$$

Here A_k are arbitrary coefficients, I_n is the Bessel function of the first kind and n-order in its imaginary argument, and p_k are the roots of the characteristic equation

$$\Delta(p) \equiv \beta p \sin p - \cos p = 0 \quad (\beta = G_0 h G^{-1}) \tag{1.6}$$

The roots of the above equation are all simple and real, and their asymptotics is given by the formula

$$p_k = \pi k + O(k^{-1}) \tag{1.7}$$

The homogeneous solutions (1.5) satisfy the following orthogonality relation (δ_{kn} is the Kronecker delta):

$$\int_0^1 \sin p_k z \sin p_n z \, dz + \beta \sin p_k \sin p_n = \delta_{kn} \gamma_k \tag{1.8}$$

$$\gamma_k = {}^1/_2 (1 + \beta \sin^2 p_k)$$

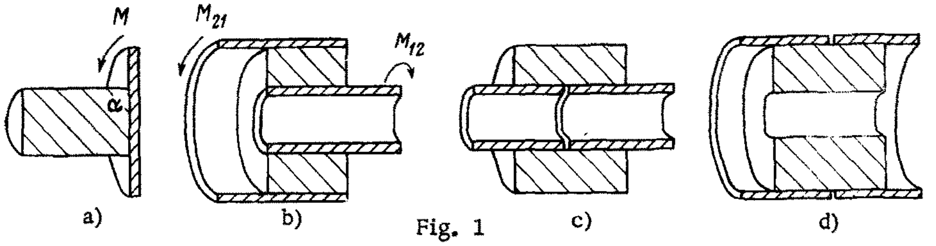

Fig. 1

The above relations together with the boundary conditions (1.3), (1.4) yield the coefficients A_k in the following manner. In accordance with (1.3), (1.5) we formally write

$$f(z) = \sum_{n=1}^{\infty} a_n \sin p_n z, \quad a_n = A_n p_n^2 I_2(p_n R) \tag{1.9}$$

Multiplying both sides of the first equation by $p_k z$, and integrating with respect to z from zero to one and using (1.8), we obtain

$$a_k = \gamma_k^{-1}\int_0^1 f(z)\sin p_k z\,dz + \beta\gamma_k^{-1}\sin p_k \sum_{n=1}^{\infty} a_n \sin p_n \tag{1.10}$$

Let us clarify the meaning of the second term in the right-hand side of (1.10). From (1.4) and (1.5) we have

$$M = 2\pi R^2\beta \sum_{n=1}^{\infty} A_n p_n^2 I_2(p_n R)\sin p_n = 2\pi R^2\beta \sum_{n=1}^{\infty} a_n \sin p_n$$

Therefore we can write (1.10) in the form

$$a_k = \gamma_k^{-1}\int_0^1 f(z)\sin p_k z\,dz + \frac{M\sin p_k}{2\pi R^2\gamma_k} \tag{1.11}$$

This means that the problem (1.1)-(1.4) has been solved in closed form. In particular, if the lateral surface of the cylinder is stress-free, then

$$A_k = M\sin p_k\,[2\pi R^2\gamma_k p_k^2 I_2(p_k R)]^{-1} \tag{1.12}$$

and the contact stresses between the cylinder and the plate can be found from (1.5),(1.6) and (1.12) using the formula

$$\tau_{\varphi z}(r,1) = \frac{M}{2\pi R^2\beta}\sum_{k=1}^{\infty}\frac{I_1(p_k r)}{p_k(1+\varepsilon_k)I_2(p_k R)} \tag{1.13}$$
$$\varepsilon_k = \frac{1+\beta}{\beta^2 p_k^2}$$

The type of behavior of the contact stresses with $r \to R$ can be easily established using the asymptotic formulas for the Bessel functions [2] and the formula (1.7). According to (1.13) we have

$$\tau_{\varphi z}(r,1) \sim \frac{M}{2\pi^2 R^2\beta}\ln\frac{1}{R-r} \quad (r\to R) \tag{1.14}$$

We note that the logarithmic singularity is unstable. If the angle α is acute, the singularity vanishes; if it is obtuse, it becomes a power singularity. This follows from the analysis of the solution of the functional difference equation [3] defining the antiplane deformation of a wedge with an elastic covering.

2. Let a finite, hollow elastic cylinder $l_1 \leqslant z \leqslant l_2$, $R_1 \leqslant r \leqslant R$ be welded by its cylindrical surfaces $r = R_j$ to shells of constant thickness h_j $(j = 1, 2)$. Axisymmetric tangential stresses $g_s(r)$ are given at the ends of the cylinder $z = l_s$ $(s = 1, 2)$ and the torsional moments M_{js} at the shell ends

(Fig. 1b). The boundary conditions of this problem have the form

$$\frac{\partial^2 v}{\partial z^2} + (-1)^{j-1} \frac{1}{G_{0j} h_j} \tau_{r\varphi} = 0 \quad (r = R_j,\ j = 1, 2) \tag{2.1}$$

$$\tau_{\varphi z}(r, l_s) = g_s(r) \quad (R_1 < r < R_2,\ s = 1, 2) \tag{2.2}$$

$$\tau_{\varphi z}(R_j, l_s) = M_{js} G \,(2\pi G_{0j} h_j R_j^2)^{-1} \quad (s = 1, 2;\ j = 1, 2) \tag{2.3}$$

where G_{0j} is the shear modulus of the j-th shell, G is the shear modulus of the cylinder and the following condition of equilibrium holds:

$$\int_{R_1}^{R_2} g_1(r) r^2\, dr + \frac{1}{2\pi}(M_{11} + M_{21}) = \int_{R_1}^{R_2} g_2(r) r^2\, dr + \frac{1}{2\pi}(M_{12} + M_{22}) \tag{2.4}$$

We write the solution in the form of a series

$$v = \sum_{k=-\infty}^{\infty} v^k, \quad v^k = A_k G^{-1} p_k e^{p_k z} Z_1(p_k, r) \quad (k = \pm 1, \pm 2, \ldots) \tag{2.5}$$

$$\tau_{\varphi z}{}^k = A_k p_k{}^2 e^{p_k z} Z_1(p_k, r), \quad \tau_{r\varphi}{}^k = -A_k p_k{}^2 e^{p_k z} Z_2(p_k, r)$$

$$v^0 = A_0 G^{-1} r z; \quad \tau_{\varphi z}{}^0 = A_0 r$$

$$Z_n(p, r) = \beta_1 p\,[Y_1(pR_1) J_n(pr) - J_1(pR_1) Y_n(pr)] - [Y_2(pR_1) J_n(pr) - J_2(pR_1) Y_n(pr)]$$

Here J_m and Y_m are m-th order Bessel functions of the first and second kind, respectively, v^k, $\tau_{r\varphi}{}^k$, $\tau_{\varphi z}{}^k$ is the system of homogeneous solutions of the problem (2.1), $\beta_j = G_{0j} h_j G^{-1}$ $(j = 1, 2)$ and the numbers p_k satisfy the characteristic equation

$$\beta_2\, p Z_1(p, R_2) + Z_2(p, R_2) = 0$$

It can be shown that all roots of the above equation are real and simple. When $\beta_j > 0$ $(j = 1, 2)$ are fixed, their asymptotics has the form

$$p_k = \pi k\,(R_2 - R_1)^{-1} + O(k^{-1})$$

According to the formulas (8) and (11) of Sect. 5.11 and (12) of Sect. 3.63 of [2], the homogeneous solutions (2.5) satisfy the following orthogonality relation:

$$\int_{R_1}^{R_2} Z_1(p_k, r) Z_1(p_n, r)\, r\, dr + \sum_{j=1}^{2} \beta_j R_j Z_1(p_k, R_j) Z_1(p_n, R_j) = \delta_{kn} \gamma_k \tag{2.6}$$

$$\gamma_k = \frac{1}{2}\left[R_2{}^2 Z_1{}^2(p_k, R_2)\left(1 + \beta_2{}^2 p_k{}^2 + \frac{4\beta_2}{R_2}\right) - \frac{4}{\pi^2 p_k{}^2}\left(1 + \beta_1{}^2 p_k{}^2 - \frac{4\beta_1}{R_1}\right)\right]$$

Substituting the series (2.5) into the condition (2.2) and utilizing the relation (2.6) with the conditions (2.3), we obtain two equations determining the coefficients A_k and A_{-k} $(k = 1,2, \ldots)$

$$p_k^2\gamma_k (A_k e^{p_k l_s} - A_{-k} e^{-p_k l_s}) = \int_{R_1}^{R_2} g_s(r) Z_1(p_k, r) r\, dr + \frac{1}{2\pi R_2} Z_1(p_k, R_2) M_{2s} + \frac{1}{\pi^2 R_1^2 p_k} M_{1s} \quad (s = 1, 2)$$

Condition of equilibrium (2.4) together with the formula (1) of Sect. 5.1 of [2] yields

$$A_0\gamma_0 = \int_{R_1}^{R_2} g_1(r) r^2\, dr + \frac{1}{2\pi}(M_{11} + M_{21})$$

$$\gamma_0 = \beta_2 R_2^3 + \beta_1 R_1^3 + {}^1/_4 (R_2^4 - R_1^4)$$

If $M_{is} \neq 0$ then $\tau_{r\varphi}(z, R_j) = O(\ln |z - l_s|)$ as $z \to l_s$, if $M_{js} = 0$, then $\tau_{r\varphi}(z, R_j) = O(1)$. We solve in the same manner other problems of torsion of thin-walled pipes connected by means of cylindrical sleeves (see, e.g. Fig. 1c and 1d). The proposed method can also be used to solve problems of simultaneous twisting of shells and finite elastic bodies bounded by conical, spherical and ellipsoidal surfaces.

3. We use the first problem to discuss the question of justifying the results obtained. This can be reduced to the examination of the series (1.9), the convergence of which implies at once the convergence of the series (1.5) for v when $r = R$; Its convergence at $r < R$ is evident.

For the time being we shall assume that the coefficients of the series (1.9) are given by the formula (1.11) with $M = 0$ and b_k are the coefficients of the Fourier sine series for the function $f(x)$

$$a_k = \gamma_k^{-1} \int_0^1 f(t) \sin p_k t\, dt, \quad b_k = \frac{1}{2} \int_0^1 f(t) \sin \pi k t\, dt$$

Let us introduce the notation

$$I = \int_a^b f(t)\, dt, \quad T_m(t, x; \beta) = \sum_{k=1}^m \sin p_k t \sin p_k x \gamma_k^{-1}$$

$$D_m(t, x) = 2 \sum_{k=1}^m \sin \pi k t \sin \pi k x, \quad S_n(t, x; \beta) = T_n(t, x; \beta) - D_N(t, x)$$

with the choice of N to be discussed later. Then

$$\int_0^1 f(t) S_n(t, x; \beta)\, dt = \sum_{k=1}^n a_k \sin p_k x - \sum_{k=1}^N b_k \sin \pi k x \tag{3.1}$$

Study of the behavior of the integral appearing in the left-hand side of the formula (3.1) enables us to relate the problem of convergence of the series (1.9) to the known results concerning the convergence of the Fourier sine series for $f(x)$. Let us use the relation

$$S_n(t, x; \beta) = \frac{1}{2\pi i}\int\limits_L \varphi(w)\,dw, \quad \varphi(w) = \frac{2\sin xw \sin tw}{\sin w\Delta(w)} \tag{3.2}$$

Where the contour L is a rectangle with vertices $(\pm Bi,\ C_n \pm Bi)$, $p_n < C_n < p_{n+1}$, $C_n \neq \pi k$, N is the integral part of $\pi^{-1}C_n$, and the function $\Delta(w)$ is given by (1.6). Since the integral along the imaginary axis, regarded as the principal value, is equal to zero, we find that for $x + t < 2$ we have

$$S_n(t, x; \beta) = \lim_{B\to\infty}\frac{1}{2\pi i}\int\limits_{C_n - Bi}^{C_n + Bi}\varphi(w)\,dw \tag{3.3}$$

The inequalities

$$|S_n(t, x; \beta)| \leqslant \frac{Q}{(2 - x - t)C_n}, \quad \left|\int\limits_0^t S_n(t, x; \beta)\,dt\right| \leqslant \frac{Q}{(2 - x - t)C_n^2}$$

which follow directly from (3.2) and (3.3) (Q is constant) yield the following analog of the Riemann-Lebesgue theorem: if for $[a, b] \subset [0, 1]$ an integral I exists and converges absolutely, then for $0 \leqslant x < 1$ (if $b < 1$, then for $0 \leqslant x \leqslant 1$) we have the estimate

$$\int\limits_a^b f(t) S_n(t, x; \beta)\,dt = o\left(\frac{1}{n}\right) \quad (n\to\infty)$$

Therefore

$$\sum_{k=1}^{n} a_k \sin p_k x - \sum_{k=1}^{N} b_k \sin \pi k x = o\left(\frac{1}{n}\right) \quad (0 \leqslant x < 1)$$

This, together with (1.7) and the Riemann-Lebesgue theorem, implies that the series (1.9) and the Fourier sine series both diverge. Consequently, the convergence (uniform convergence) of the Fourier sine series to the function $f(x)$ represents a sufficient condition for the convergence of the series (1.9) to $f(x)$ for $0 \leqslant x < 1$ (uniform convergence for $0 \leqslant x \leqslant 1 - \delta$, $\delta > 0$). In particular if $f(x) \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$, then $b_n = O(n^{-\alpha})$ [4]

$$\sum_{k=1}^{n} b_k \sin \pi k x - f(x) = O(n^{-\alpha}\ln n)$$

and the deviation of the partial sums of the series (1.9) from the function $f(x)$ is of the same order.

Let us now consider the problem of convergence of (1.9) near the point $x = 1$. Analogously with (3.2) and (3.3) we obtain

$$\int_0^1 f(t)\, T_n(t, x; \beta)\, dt = \sum_{K=1}^{n} a_k \sin p_k x$$

$$T_n(t, x; \beta) = \lim_{B\to\infty} \frac{1}{2\pi i} \int_{C_n - Bi}^{C_n + Bi} K(w, x, t)\, dw$$

$$K(w, x, t) = \begin{cases} \psi(w, x) \sin tw, & 0 \leqslant t < x \leqslant 1 \\ \psi(w, t) \sin xw, & 0 \leqslant x < t \leqslant 1 \end{cases}$$

$$\psi(w, x) = \frac{2}{\Delta(w)} [\beta w \sin(x-1) w + \cos(x-1) w]$$

Using the estimate $\quad (0 \leqslant t \leqslant 1, \quad 0 \leqslant x \leqslant 1)$

$$\left| \int_0^t T_n(t, x; \beta)\, dt \right| = O(1)$$

$$\left| \int_0^t T_n(t, 1; \beta)\, dt \right| = O\left(\frac{1}{C_n}\right) \qquad (n \to \infty)$$

we can show that if the integral I, exists and converges absolutely, the function $f(x)$ is continuous and has a bounded variation on $(\lambda, 1)\ (0<\lambda<1)$ and $f(1) = 0$ then the series (1.9) converges uniformly near $x = 1$.

Let us now set in (1.11) $M \neq 0$ and $f(z) \equiv 0$. Then the series (1.9) converges to zero when $0 \leqslant x < 1$ and to M when $x = 1$. This can be easily confirmed by considering the complex integrals

$$\int_{L_1} \frac{(\beta+1)\sin w + \beta w \cos w}{\Delta(w)(1+\beta+\beta^2 w^2)}\, dw, \quad \int_{L_2} \frac{\sin xw}{\Delta(w)}\, dw$$

where the contour L_1 is the circumference of radius C_n with the center at the coordinate origin and L_2 is a rectangle with the vertices $(\pm\pi(n + {}^1/_2),\ \pm\ln n)$.

4. Solution of problems shown schematically in Fig. 1 was obtained using the digital computer M-222. Below we give some of the results. Fig. 2 refers to the problem (1.1)-(1.4) with $R = 1, f(z) \equiv 0, M = 2\pi$ and various values of β: the graphs depict the distributions of the contact stresses $\tau_{\varphi z}(1, r)$ and forces $\beta\tau_{r\varphi}(1, r)$ in the plate. Figs. 3 and 4 refer to the problem (2.1)-(2.3) with $R_1 = 1, R_2 = 2$, $l_1 = 0,\ l_2 = 2,\ g_s(r) \equiv 0,\ M_{ss} = 0\ (s = 1, 2),\ M_{21} = M_{12} = 1,\ \beta_1 = \beta_2 = \beta$ and various β. The graphs of the functions $N_j = R_j\tau_{r\varphi}(z, R_j)$ in the left half of Fig. 3 correspond to the outer contact surface $(j = 2)$. Fig. 4 depicts the graphs of the moments of tangential forces $M_j = 2\pi R_j^2 \beta_j \tau_{\varphi z}(z, R_j)$ in the shells. The family of curves passing through the coordinate origin corresponds to the inner shell, and the other family to the outer shell. Dashed lines depict limiting solutions of the problem in question for a plane stamp and perfectly rigid cylindrical yokes.

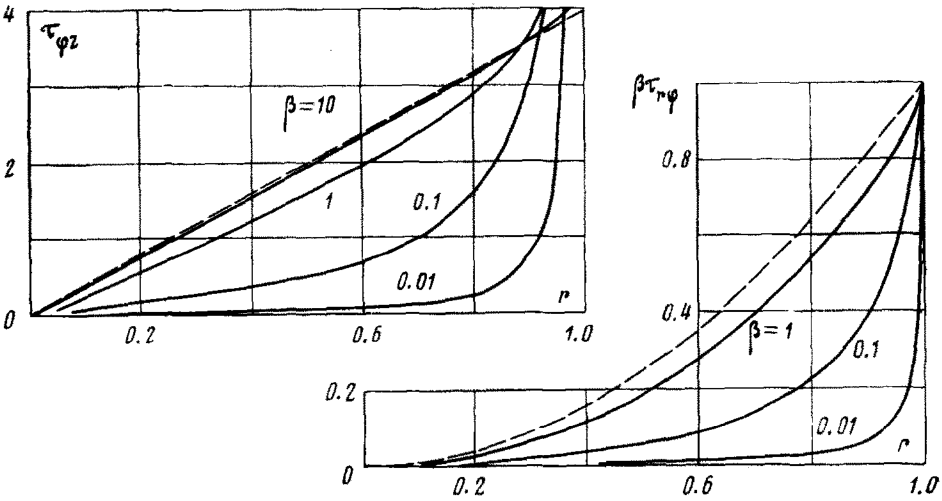

Fig. 2

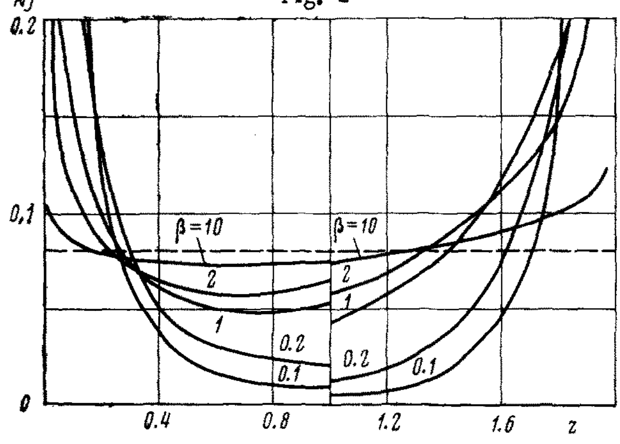

Fig. 3

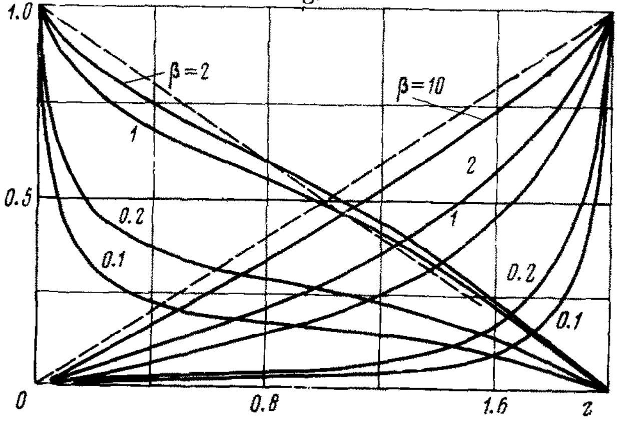

Fig. 4

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